# Supertubes in matrix model and DBI action

**Seiji Terashima**

*Yukawa Institute for Theoretical Physics, Kyoto University*
*Kyoto 606-8502, Japan*
*E-mail:* `terasima@yukawa.kyoto-u.ac.jp`

ABSTRACT: We show the equivalence between the supertube solutions with an arbitrary cross section in two different actions, the DBI action for the D2-brane and the matrix model action for the D0-branes. More precisely, the equivalence between the supertubes in the D2-brane picture and the D0-brane picture is shown in the boundary state formalism which is valid for all order in $\alpha'$. This is an application of the method using the infinitely many D0-branes and anti-D0-branes which has been used to show other equivalence relations between two seemingly different D-brane systems, including the D-brane realization of the ADHM construction of instanton. We also apply this method to the superfunnel type solutions successfully.

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## 1. Introduction and summary

There are (marginal) bound states of the D-branes with different dimensions in string theory, which have been very important for studying string theory. Furthermore, it have been known that such bound states have two (or several) different, but equivalent descriptions. The typical example is the bound state of the D4-branes and the D0-branes [1, 2]. For a bound state of D$p$-branes and D$q$-branes, we can think the D$p$-branes as solitons in the D$q$-branes. Instead, we can think the D$q$-branes as solitons in D$p$-branes. This equivalence or duality between different D-brane systems is one of the characteristic properties of the D-branes and will be important to investigate further. However, it is in general very difficult to prove the duality although the evidences for the duality have been given for many bound states of D-branes by using the low energy effective actions and supersymmetry. The two dual D-brane systems are very different in their low energy effective actions, for example, the DBI action and the matrix model action. Therefore, it will be impossible to show the duality by using the low energy effective actions. Moreover, these two actions are only valid for different regions in the parameter space except in some large $N$ limit

where $N$ is the number of D-branes. Then we should heavily rely on the BPS protected properties to connect two different regions of the parameter space. Using the boundary state formalism the duality can be shown, however, only for some special cases [3, 4].

Recently, a unified picture of such dualities was given by the tachyon condensation of the unstable D-branes in the boundary state formalism [5] [1] and successfully applied to the bound state of the D4-brane and the D0-branes, i.e. ADHM(N) construction of instantons and monopoles [6, 7]. In this paper, we will consider this duality for the supertubes by the method using the tachyon condensation [5].

The supertubes are the 1/4 supersymmetric bound states of the D0-branes with fundamental strings which are expanded to the tubular D2-branes [8]. Therefore the supertubes have two different pictures: the D0-branes and the D2-brane. In the D0-brane picture the supertube solutions was found in the matrix model by [9] and the supertubes have been investigated extensively in both pictures [10]-[17].

In this paper we construct the supertube solution of the D2-brane from the infinitely many D0-branes and anti-D0-branes. Then by using the different basis of the Chan-Paton factor of the D0$-\overline{\text{D0}}$-branes, we find the supertubes in the D0-brane picture. This implies that these two supertubes solutions in the D2-brane picture and the D0-brane picture are indeed equivalent in the boundary state formalism which is valid for all order in $\alpha'$.[2]

Some remarks are as follows. In this paper, we will consider the flat spacetime and the classical limit $g_s \to 0$ only, thus we will drop the $g_s$ dependence for notational simplicity. We will show the equivalence between the boundary states which can be off-shell. The on-shell conditions can be derived in some approximations, including the DBI actions and the matrix model actions.

Organisation of this paper is as follows. In section 2 we review how to obtain the duality or equivalence relations between two different D-brane system from the D-brane-anti-D-brane system. Section 3 is for the brief review of the supersymmetric solitons with electric field. Then the method explained in section 2 is applied to the supertubes with compact cross section in section 4 and with non-compact cross section in section 5. The superfunnel type solutions are also discussed in section 5. In the appendix we show the duality for the non compact supertube from the D2-brane-anti-D2-brane system.

## 2. Duality between D0-branes and Dp-brane via tachyon

In this section we will review how to obtain the duality between Dq-branes and Dp-brane using the Dp-branes and the anti Dp-branes [5], in particular, the duality between D0-branes and Dp-brane in type IIA string theory using the D0$-\overline{\text{D0}}$-brane pairs, which will be applied to the supertubes.

### 2.1 Diagonalization of the tachyon and the duality

Consider $N$ D$p$-branes and $N'$ anti-D$p$-branes. They have the tachyon $T'$ which is a complex

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[1]This can be regarded as a generalization of [3].

[2]The boundary state of the supertube in the T-dual of D0-brane picture was found in [18] although we will not use it.

$N \times N'$ matrix. Note that the Chan-Paton bundle of the $N$ D$p$-branes is an $N$ dimensional vector space on which $T'$ acts. For the $N'$ anti-D$p$-branes, there is an $N'$ dimensional vector space on which $T'^{\dagger}$ acts. Any orthonormal basis of the two vector spaces can be used and the basis change is the $U(N) \times U(N')$ gauge symmetry on the D-branes and the anti-D-branes. Then, generally, by giving topologically nontrivial expectation value to the tachyon, we get the D$q$-branes with $q \neq p$, i.e. the decent relation [19, 20] for $q < p$ or the accent relations for $q > p$ [5, 21]. Thus after the tachyon condensation we can identify the system of the D$p$-branes and the anti-D$p$-branes as the D$q$-branes (with surviving D$p$-branes or anti-D$p$-branes after the tachyon condensation).

Now let us choose another basis of the Chan-Paton bundle in which the tachyon $T'$ is diagonal. In this gauge, we expect that the D$p$-branes (and the anti-D$p$-branes) corresponding to the nonzero eigen value of the $T'$ (and $T'^{\dagger}$) disappear, respectively. Then, only the D$p$-branes (or the anti-D$p$-branes) corresponding to the zero modes of $T'$ (or $T'^{\dagger}$) will remain after the tachyon condensation, respectively. Actually, in the boundary state formalism or the boundary string field theory, this is justified [22–24]. Therefore the system of the D$p$-branes and the anti-D$p$-branes with the nontrivial tachyon condensation is equivalent to D$p$-branes or anti-D$p$-branes corresponding to the zero modes of the tachyon. Note that we had the two different results of the same tachyon condensation with the different gauge choices.

Thus, combining these two equivalences, we have an equivalence or a duality between the the D$q$-branes and D$p$-branes (or anti-D$p$-branes corresponding to the zero modes) [5]. This was explicitly performed for the flat noncommutative D-brane [25, 5], fuzzy sphere [5] and ADHM(N) construction of the instantons and monopoles [6, 7]. In the next subsection we will explain this duality for $p = 0$ case in detail.

## 2.2 Duality between infinitely many D0-branes and D$p$-brane

Consider $N$ D0$-\overline{\text{D0}}$-brane pairs and let a D0-brane and an anti-D0-brane of each pair are at a same position. Then, the fields on the pairs are $N \times N$ matrix coordinates $X^{\mu}$ and the complex tachyon $T'$, other than massive fields which are not relevant in this paper. Instead of $T'$, we will use $2N \times 2N$ Hermitian matrix $T = \begin{pmatrix} 0 & T' \\ T'^{\dagger} & 0 \end{pmatrix}$ for convenience.

Now, let us consider a large $N$ limit, then the matrices $X^{\mu}$ and $T$ become operators acting on a Hilbert space. The important configuration of the $N$ D0$-\overline{\text{D0}}$-brane pairs is

$$T = u\Gamma^{i}(\partial_{i} + iA_{i}(x)), \quad \hat{X}^{i} = x^{i}, \quad (i = 1, \cdots, p) \tag{2.1}$$

where the Hilbert space is spanned by the (normalizable) spinors on $\mathbf{R}^{p}$, $\Gamma^{i}$ is the gamma matrix on $\mathbf{R}^{p}$, $A_{i}(x)$ is a arbitrary function of $x^{j}$ and $u$ is just a constant which is sent to infinity. In the $u \rightarrow \infty$ limit, the configuration (2.1) represents a flat D$p$-brane with the background gauge field $A_{i}(x^{i})$ and it becomes the exact solution of the equations of motion of the boundary string field theory for the constant field strength [26, 21, 27]. (If $u$ is finite, (2.1) does not corresponds to the solution of the equations of motion and is expected to represent the D$p$-brane with some off-shell excitations.) This is "T-dual" to the well

known decent relation [20], which state that we can have a D$q$-brane by the nontrivial tachyon condensation on D$p$ − $\overline{\text{D}p}$-brane pairs where $q < p$.[3] Moreover, we can give the expectation value to the remaining transverse scalars, $\hat{X}^a$, on the D0−$\overline{\text{D0}}$-branes which should be considered as an operator acting on the spinors: $\hat{X}^a = \phi^a(x^i)$ $(a = p+1, \cdots, 9)$ where $\phi^a(x^i)$ is an arbitrary function of $x^i$. Then the D$p$-brane has the transverse scalars $X^a = \phi^a(x^i)$.

Here it is important to note that the tachyon is proportional to the Dirac operator on the world volume of the D$p$-brane,

$$T = u\,\slashed{D}, \tag{2.2}$$

and the D0-branes and anti-D0-branes correspond to the spinors with positive and negative chirality respectively.

Now assuming the "Hamiltonian" $\mathcal{H} \equiv \slashed{D}^2$ has a gap above the ground states or zero modes

$$\mathcal{H}\,|\,a\,\rangle = 0, \quad (a = 1, \cdots, n) \tag{2.3}$$

where we define the number of the zero modes as $n$. Then we can follow the argument in the previous subsection: Because $T^2 = u^2\slashed{D}^2 \to \infty$ for nonzero modes, the tachyon condensation forces D0−$\overline{\text{D0}}$-brane pairs corresponding to the nonzero modes disappear and only the D0-branes and the anti-D0-branes corresponding to the zero modes survive. Then the $n \times n$ matrix coordinates for the zero modes, $\hat{X}_0^\mu$, will become

$$\left(\hat{X}_0^\mu\right)_{ab} = \langle\,a\,|\,\hat{X}^\mu\,|\,b\,\rangle, \quad (\mu = 1, \cdots, 9) \tag{2.4}$$

which is in general non-commutative although the infinite dimensional matrix $\hat{X}^i(= x^i)$ and $\hat{X}^a$ are commutative each other. Note that we have only D0-branes or only anti-D0-branes depending on the sign of the background field strength on the D$p$-brane, for a generic $T$, i.e. all the zero modes are either positive chirality or negative chirality. We will consider these cases only below. Thus we have the equivalence or duality between the D$p$-brane with the back ground gauge field $A_i$ and the $n$ D0-branes (or anti-D0-branes) with the coordinates $\hat{X}_0^\mu$.

It is easy to consider the D$p$-brane with a curved world volume which is specified by the embedding $X^\mu = X^\mu(x^i)$. Taking the tachyon as a Dirac operator with the vielbein consistent with the induced metric, we have the D$p$-brane after the tachyon condensation [21, 5]. Then, repeating the procedures for the flat D$p$-brane, i.e. computing the matrix elements of the $\hat{X}^\mu$ between the zero-modes of the Dirac operator, we have the D0-brane picture which should be equivalent to the D$p$-brane picture.

It is important to note that all the results in this paper can be given and justified in the boundary state formalism [28, 5] where the fields, like the tachyon, are represented in the world sheet boundary action, although we will not explicitly write the corresponding

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[3]Both the decent relation and the accent relation (2.1) are obtained from $[\mathcal{D}_i, T] = u\Gamma^i$ and $[\mathcal{D}_i, \mathcal{D}_j] = 0$ by some truncation. However, this is not the T-dual each other, of course. The accent relation is also regarded as a generalization of the matrix model construction of a D($2p$)-brane using the noncommutative coordinates.

boundary states.[4] Thus, the equivalences given in this paper are exact in all order in $\alpha'$.[5] We also note that the boundary state includes any information about the D-brane, thus the equivalences in this paper imply the equivalences between the tensions, the couplings to the closed strings and the D-brane charges, of the D$p$-brane and the D0-branes.[6]

## 3. Supersymmetric Solitons with electric field

Let us consider the BFSS matrix model [29] which is same as the "low energy" effective action of the $N$ D0-branes in type IIA theory. The equation of motion

$$-\mathcal{D}_t^2 \hat{X}^\mu + [\hat{X}_\nu, [\hat{X}^\nu, \hat{X}^\mu]] = 0 \quad (\mu = 1, \ldots, 9) \tag{3.1}$$

and Gauss low constraint

$$[\hat{X}_\mu, \mathcal{D}_t \hat{X}^\mu] = 0 \tag{3.2}$$

are simplified for time independent configurations with

$$\hat{A}_t = \pm \hat{Z}, \quad (\hat{Z} \equiv \hat{X}^9) \tag{3.3}$$

to

$$[\hat{X}_i, [\hat{X}^i, \hat{X}^\mu]] = 0 \quad (i = 1, \ldots, 8). \tag{3.4}$$

From this form, we can easily find two types of solitons: the supertube type and the superfunnel type.

Introducing the operators $\hat{a}$ and $\hat{N}$ acting on the space spanned by the ket $|m\rangle$ such that

$$\hat{a}|m\rangle = |m-1\rangle, \quad \hat{N}|m\rangle = m|m\rangle, \tag{3.5}$$

where $m \in \mathbf{Z}$, the supertube solution [16] is given by

$$\hat{Z} = \hat{N} - \varphi, \quad \hat{X}^i = \hat{X}^i(\hat{a}, \hat{a}^\dagger), \tag{3.6}$$

where $X^i(a, a^\dagger)$ is an arbitrary functions of $a, a^\dagger$ and $0 \leq \varphi < 1$. We can check this is the solution of (3.4) by using the following relations derived from the definition (3.5):

$$[\hat{N}, \hat{a}] = -\hat{a}, \quad \hat{a}^\dagger = \hat{a}^{-1}, \quad [\hat{a}^\dagger, \hat{a}] = 0, \quad [\hat{N}, f(\hat{a})] = -\frac{\partial f(\hat{a})}{\partial \ln \hat{a}}, \tag{3.7}$$

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[4]We will use the naive off-shell extension of the boundary state, i.e. just inserting the possibly non-conformal world sheet boundary action to the (on-shell) boundary state. In this paper, the boundary state means such a naively extended boundary state.

[5]In this paper we take $\alpha' = 2$.

[6]In particular, the equivalence between the D0-brane charges are reduced to the index theorem [28]. Actually, the three representations of the index

$$\int_{D_p} \operatorname{Tr} e^F \hat{\mathcal{A}}, \quad \operatorname{Tr}_{\text{zero modes of } \not{p}}(\Gamma), \quad \operatorname{Tr}\left(\Gamma e^{-u^2 \not{p}^2}\right), \tag{2.5}$$

correspond to the D2-brane, the D0-branes and the D0$-\overline{\text{D0}}$-brane pictures, respectively.

where $f(a)$ is an arbitrary function of $a$. Note that $\hat{a}^\dagger = \hat{a}^{-1}$ implies $\hat{a}^\dagger \hat{a} = 1$. For the circular case, i.e. $\hat{X}^1 + i\hat{X}^2 = R\,\hat{a}$ and others$= 0$, this is the solitons obtained in [9]. For a general profile, we will show later that this indeed represents the supertube with a general cross section.[7]

The other type of solution is given by

$$\hat{Z} = \hat{Z}(\hat{X}^b), \quad [\hat{X}^b, \hat{X}^c] = i(1/B)^{bc}, \quad (b, c = 1, \dots, 2p) \tag{3.8}$$

with

$$[\hat{X}_b, [\hat{X}^b, \hat{Z}(\hat{X}^c)]] = 0. \tag{3.9}$$

If we take $B_{2i-1,2i} = -B_{2i,2i-1} = b_i$ $(i = 1, \dots, p)$ and others$= 0$, the eq. (3.9) becomes

$$\triangle \hat{Z}(\hat{X}) \equiv \sum_{i=1}^{p} \frac{1}{b_i^2} \left( \frac{\partial^2}{\partial^2 \hat{X}^{2i-1}} + \frac{\partial^2}{\partial^2 \hat{X}^{2i}} \right) \hat{Z}(\hat{X}) = 0, \tag{3.10}$$

where $\hat{Z}(\hat{X})$ is Weyl ordered. Here we have used the fact that the derivative of the Weyl ordered product is again Weyl ordered. Using the $\triangle G(X^i) = \delta(X^i)$, we can write the solution as

$$\hat{Z}(\hat{X}^i) = Z_0 + \sum_a Q_a G(\hat{X}^i - x_a^i), \tag{3.11}$$

where $Q_a$ will be quantized by the flux quantization condition. For $p = 1$, this solution corresponds to the superfunnel of [17]. In particular for $\hat{Z}(\hat{X}^1, \hat{X}^2) = Q/2 \log((\hat{X}_1)^2 + (\hat{X}_2)^2)$, this solution have been obtained in [16, 30] explicitly. For a general $p$, this corresponds to the BIon [31, 32] with magnetic field. Actually, it is well-known that by expanding around $\hat{Z} = 0$ and $[\hat{X}^i, \hat{X}^j] = i\Theta_{ij}$, the action of the D0-branes becomes the noncommutative D($2p$)-brane action with magnetic fields [33]. Thus the solution (3.11) corresponds to the noncommutative BIon.

Note that we could consider a combination of the supertube type and the superfunnel type of the solutions.

We note that for the D$p$-branes instead of D0-branes the equations of motion (3.4) is valid if we think $X_k$ as $\mathcal{D}_k \equiv \partial_k + A_k$ for $k = 1, \dots, p$. and take $\partial_t + iA_0 = iZ$.

Now, following [16], we will discuss the BPS equations of the matrix model. The supersymmetric variation of the fermions in the matrix model is

$$\delta\psi = \left( \mathcal{D}_t \hat{X}^\mu \gamma_\mu + \frac{i}{2}[\hat{X}^\mu, \hat{X}^\nu]\gamma_{\mu\nu}\epsilon + \epsilon' \right). \tag{3.12}$$

For the time independent configurations with $\hat{A}_t = \hat{Z}$, we have the BPS equations

$$\delta\psi = \left( i[\hat{X}^i, \hat{X}^j]\gamma_{ij}P_-\epsilon + \epsilon' \right), \tag{3.13}$$

where we have defined the real projection operator $P_\pm = (1 \pm \gamma_z)/2$ and set $P_+\epsilon = 0$. For $[\hat{X}^i, \hat{X}^j] = i\Theta^{ij}$ where $\Theta^{ij}$ is a constant, we can solve it by $\epsilon' = -i[\hat{X}^i, \hat{X}^j]\gamma_{ij}P_-\epsilon$. Therefore, the time-independent solution of the Gauss law constraint (3.9) with $\hat{A}_t = \hat{Z}$ and $[\hat{X}^i, \hat{X}^j] = i\Theta^{ij}$ is indeed the 1/4 BPS solution of the matrix model.

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[7]For a non-compact world volume, the supertube is given by $\hat{Z} = \hat{z}$ and $\hat{X}^i = \hat{X}^i(\hat{x})$ where $[\hat{x}, \hat{z}] = i/B$ and $B$ is a constant.

## 4. Supertube and the duality

### 4.1 Circular supertube

In this subsection we consider a supertube of a circular cross section which is a simplest supertube of a compact cross section in the flat spacetime. In particular, we consider a D2-brane in type IIA superstring theory whose world volume coordinates are labelled by $(t, z, \theta)$ with a constant field strength $F = Edt \wedge dz + Bdz \wedge d\theta$. We will take a gauge,

$$A_z = 0, \ \ A_t = -Ez, \ \ A_\theta = Bz + \varphi, \tag{4.1}$$

where $\varphi$ is a Wilson line which can be taken as $0 \leq \varphi < 1$.[8] Although we have taken a particular gauge, the Wilson line has a gauge invariant and physical meaning. The configuration of the D2-brane in the flat spacetime is $X^0 = t, X^1 = R\cos\theta, X^2 = R\sin\theta, X^3 = z$ and $X^i = 0$ for $i = 4, \ldots, 9$ and the induced metric on the D2-brane is given by $ds^2 = -dt^2 + dz^2 + R^2 d\theta^2$. If we take $E = \pm 1$, this configuration becomes BPS and satisfies the classical equations of motion of the DBI action [8].

Now we will represent this D2-brane configuration as a soliton of infinitely many $D0-\overline{D0}$-brane pairs as explained in the subsection 2.2. The tachyon of the $D0-\overline{D0}$-brane pairs is

$$T = u\slashed{D}, \tag{4.2}$$

where $\slashed{D}$ is the Dirac operator on a time slice of the world volume of corresponding D2-brane (i.e. on the $\mathbf{R} \times S^1$),

$$i\slashed{D} = \sigma_1 \frac{1}{R}\left(\frac{\partial}{\partial\theta} - i(Bz + \varphi)\right) + \sigma_2 \frac{\partial}{\partial z}. \tag{4.3}$$

The transverse scalars of the D0-anti D0-brane pairs are given by

$$\hat{X}^1 = R\cos\theta, \ \hat{X}^2 = R\sin\theta, \ \hat{X}^3 = z, \ \hat{X}^i = 0 \ \ (i = 4, \ldots, 9), \tag{4.4}$$

and

$$\hat{A}_t = -Ez, \tag{4.5}$$

where the tachyon, the scalars and the gauge potential, which are $N \times N$ matrices for the $N$ pairs, become operators acting on the Hilbert space spanned by the normalizable spinor valued functions on the $\mathbf{R} \times S^1$ parametrised by $\{z, \theta\}$ by taking a particular large $N$ limit. Note that the information of the gauge potential of the D2-brane except $\hat{A}_t$ are encoded in the tachyon of the D0-anti D0-brane pairs. Then, these $D0-\overline{D0}$-brane pairs are equivalent to the supertube of the D2-brane (in the $u \to \infty$ limit) [26, 21].

In order to obtain the matrix model picture (or more precisely the D0-brane picture), we need to find the zero modes of the Dirac operator in the $D0-\overline{D0}$-brane picture, which corresponds to the D0-branes remain after the tachyon condensation. Thus what we should solve is

$$\slashed{D}\Psi(\theta, z) = 0. \tag{4.6}$$

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[8]Note that the space-time is $R^{1,9}$ and the embedding of the $S^1$ associated to the Wilson line into the space-time is contractable. Thus, even if we compactify the space-time to a tori in $X^i$ directions, this Wilson line does not corresponds to the position in the T-dual picture.

Because $\theta$ is periodic, we expand $\Psi$ as $\Psi(\theta, z) = \sum_{m=-\infty}^{\infty} e^{im\theta} \tilde{\Psi}_m(z)$. Then we have

$$\left( 1_{2\times 2} \frac{1}{R} (m - Bz - \varphi) + \sigma_3 \frac{\partial}{\partial z} \right) \tilde{\Psi}_m(z) = 0, \tag{4.7}$$

where we multiplied $-i\sigma_1$ to the zero-mode equation (4.6). Then for $B > 0$ we find the zero-modes $\Psi_m(\theta, z)$ parametrised by an integer $m$,

$$\Psi_m(\theta, z) = \left( \frac{R}{4\pi B} \right)^{\frac{1}{4}} \exp\left( -\frac{B}{2R} \left( z - \frac{m-\varphi}{B} \right)^2 - im\theta \right) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{4.8}$$

and for $B < 0$,

$$\Psi_m(\theta, z) = \left( \frac{R}{4\pi |B|} \right)^{\frac{1}{4}} \exp\left( -\frac{|B|}{2R} \left( z - \frac{m-\varphi}{B} \right)^2 - im\theta \right) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{4.9}$$

where $\Psi_m$ is normalized such that,

$$\int d\theta dz \Psi_m^\dagger \Psi_n = \delta_{mn}. \tag{4.10}$$

Note that the chirality operator $i\sigma_1\sigma_2 = -\sigma_3$ distinguish the D0-brane and the anti-D0-brane. We will consider $B > 0$ case below without loss of generality. Then the matrix elements of $\hat{X}^\pm \equiv \hat{X}^1 \pm i\hat{X}^2 = Re^{\pm i\theta}, \hat{X}^3 = z$ and $\hat{A}_t = -Ez$ between the zero-modes are computed as

$$(\hat{X}_0^+)_{mn} = \int d\theta dz \Psi_m^\dagger(\theta, z) Re^{i\theta} \Psi_n(\theta, z) = Re^{-\frac{1}{4BR}} \delta_{n,m+1}, \quad (\hat{X}_0^-)_{mn} = Re^{-\frac{1}{4BR}} \delta_{n+1,m},$$

$$(\hat{X}_0^3)_{mn} = \int d\theta dz \Psi_m^\dagger(\theta, z) z \Psi_n(\theta, z) = \frac{n-\varphi}{B} \delta_{m,n},$$

$$(\hat{A}_{t,0})_{mn} = -E (\hat{X}_0^3)_{mn}. \tag{4.11}$$

Therefore, in the $u \to \infty$ limit, we have the D0-branes parametrised by an integer $m$ with the matrix coordinates $(\hat{X}_0^i)_{mn}$ and a gauge field $(\hat{A}_{t,0})_{mn}$. As explained in the previous section, this system of the infinitely many D0-branes should be equivalent to the D2-brane. Note that from (4.11) we can see that the $n$-th D0-brane is at $z = \frac{n-\varphi}{B}$, however, it does not have a definite position for the $x$ and $y$ directions because of the noncommutativity except in the $B \to \infty$ limit. The nonzero Wilson line in the D2-brane picture corresponds to the shift of the D0-branes along the $z$ axis in the D0-brane picture. We also see that the density of D0-branes along $z$ is $B$ as expected. Note that $E$ has mass dimension 2, but $B$ has mass dimension 1 from the definition $F = Edt \wedge dz + Bdz \wedge d\theta$.

Introducing the operators $\hat{a}$ and $\hat{N}$ acting on the space spanned by the ket $|m\rangle$ such that

$$\hat{a} |m\rangle = |m-1\rangle, \quad \hat{N} |m\rangle = m |m\rangle, \tag{4.12}$$

where $|m\rangle$ corresponds to the remaining $m$-th D0-brane, we can view the matrices as the operators acting on this space:

$$\hat{X}_0^+ = Re^{-\frac{1}{4BR}} \hat{a}^\dagger, \quad \hat{X}_0^- = Re^{-\frac{1}{4BR}} \hat{a}, \quad \hat{X}_0^3 = \frac{\hat{N} - \varphi}{B}, \tag{4.13}$$

which satisfy

$$[\hat{X}_0^3, \hat{X}_0^\pm] = \pm\frac{1}{B}\hat{X}_0^\pm, \quad [\hat{X}_0^+, \hat{X}_0^-] = 0. \tag{4.14}$$

From the relation following from (3.7), $\hat{a}^\dagger\hat{a} = 1$, we find

$$\hat{X}_0^+\hat{X}_0^- = (\hat{X}_0^1)^2 + (\hat{X}_0^2)^2 = R^2 e^{-\frac{1}{2BR}}, \tag{4.15}$$

is the Casimir.

We note that although the zero-mode $\Psi_m(\theta, z)$ was identified as $|m\rangle$, in the reduced Hilbert space we can also represent $|m\rangle$ as $e^{i\theta m}$ and $\{\hat{a}, \hat{N}\}$ as $\{e^{-i\theta}, \frac{1}{i}\frac{\partial}{\partial\theta} - \varphi\}$.

It is very interesting to note that (4.14) solves the equations of motion of the matrix model for the BPS D0-branes. Actually, this is same as the solution of the matrix model equations of motion given in [9] although the parametrisation of $R$ is different. Therefore we have shown the equivalence between the supertube of the D2-brane picture and the supertube of the D0-brane picture.

In the D2-brane picture we took the solution of the BPS equations of the DBI action. Because the DBI action neglects the higher derivative terms and the matrix model action neglects the higher order terms, the BPS equations of the DBI action should be different from the BPS equations of the matrix model. Thus we expect the supertube solution of the DBI action is BPS even if we include $\alpha'$ corrections since the both BPS equations are satisfied at the same time.

We have seen that the D0-branes with (4.11) is essentially same as the solution of the matrix model of [9] which was supposed to be the same as the D2-brane. Here we stress that the correspondence between the D2-brane and the D0-branes are exact in our approach. One of the important points of the correspondence is that the naively defined radius of the supertube are not same for the two equivalent systems. Actually, the square root of (4.15) could be considered as the radius of the supertube, but it is smaller than $R$ by a factor of $\exp(-\frac{1}{2BR})$. We will discuss the meanings of this difference in the subsection 4.3.

## 4.2 General cross section

Let us consider general embedding of the circular supertube to the target space:

$$X^j = \sum_{m=-\infty}^{\infty} (\alpha_m^j e^{-im\theta} + \bar{\alpha}_m^j e^{im\theta}), \quad X^0 = t, \quad Z \equiv X^9 = z, \tag{4.16}$$

where $j = 1, 2, \cdots, 8$. Let $2\pi R$ as the circumference of the embedding circle in space-time. Note that $\alpha_m^j = \mathcal{O}(R)$ and for the circular supertube, $\alpha_1^1 = \alpha_1^2 = R/2$, others= 0. Though the vielbeins (and the induced metric) will generically depends on $\theta$, we can redefine the coordinate $\theta$ ($0 \le \theta < 2\pi$) such that

$$i\slashed{D} = \frac{1}{R}\sigma_1\left(\frac{\partial}{\partial\theta} - iA_\theta\right) + \sigma_2\frac{\partial}{\partial z}, \tag{4.17}$$

i.e. $(dX^j)^2 = R^2 d\theta^2$. Now we consider the D2-brane with (4.17) and (4.16). We take

$$A_\theta = Bz + \varphi, \quad A_t = -Ez, \quad A_z = 0, \tag{4.18}$$

where $B$ and $\varphi$ is constant in order to obtain the zero-modes easily although $B$ can depend on $\theta$ for assuring the supertube is a BPS state (at least in the DBI approximation) [12].

Then the configuration of the D0$-\overline{\text{D0}}$-brane pairs for this D2-brane is given by $T = u\slashed{D}$, $\hat{X}^j$ and $\hat{Z}$ corresponding to (4.16) and $\hat{A}_t = -EZ$. Because the Dirac operator has same form as (4.3), we already know the zero-modes of the Dirac operator. Thus we find

$$
\begin{aligned}
(\hat{Z}_0)_{mn} &= \int d\theta dz \Psi_m^\dagger(\theta, z) z \Psi_n(\theta, z) = \frac{n - \varphi}{B} \delta_{m,n}, \\
(\hat{A}_{t,0})_{mn} &= -E \, (\hat{Z}_0)_{mn},
\end{aligned}
\tag{4.19}
$$

and what we should do is to compute the matrix elements of $\hat{X}^j$ between the zero-modes. We can easily compute it:

$$
\begin{aligned}
(\hat{X}_0^j)_{mn} &= \int d\theta dz \Psi_m^\dagger(\theta, z) \left( \sum_{k=-\infty}^\infty (\alpha_k^j e^{-ik\theta} + \bar{\alpha}_k^j e^{ik\theta}) \right) \Psi_n(\theta, z) \\
&= \sum_{k=-\infty}^\infty \exp\left( -\frac{k^2}{4BR} \right) (\alpha_k^j \delta_{n,m-k} + \bar{\alpha}_k^j \delta_{n,m+k}).
\end{aligned}
\tag{4.20}
$$

Thus, on the Hilbert space $\{|m\rangle\}$ we represent those as

$$
\begin{aligned}
\hat{X}_0^j &= \tilde{X}^j(\hat{a}^\dagger) \\
\hat{Z}_0 &= = \frac{\hat{N} - \varphi}{B}, \\
\hat{A}_{t,0} &= -E \frac{\hat{N} - \varphi}{B},
\end{aligned}
\tag{4.21}
$$

where

$$
\tilde{X}^j(e^{i\theta}) \equiv \sum_{k=-\infty}^\infty e^{-\frac{k^2}{4BR}} (\alpha_k^j e^{-ik\theta} + \bar{\alpha}_k^j e^{ik\theta}).
\tag{4.22}
$$

Note that if we take the base of the Hilbert space as $e^{im\theta} = |m\rangle$, we have

$$
\hat{X}_0^j = \tilde{X}^j(e^{i\theta}), \quad \hat{Z}_0 = -i\frac{1}{B}\frac{\partial}{\partial \theta} - \frac{\varphi}{B} = -\frac{1}{E}\hat{A}_{t,0}.
\tag{4.23}
$$

Here we introduce the coordinate eigen state

$$
|\theta\rangle = \sum_m e^{im\theta} |m\rangle,
\tag{4.24}
$$

which satisfies $\hat{a}|\theta\rangle = e^{i\theta}|\theta\rangle$ and $\hat{a}^\dagger|\theta\rangle = e^{-i\theta}|\theta\rangle$. Then, we have the following relations

$$
\langle \theta | A(a^\dagger) | \theta' \rangle = 2\pi\delta(\theta - \theta')A(e^{i\theta})
\tag{4.25}
$$

and

$$
\text{Tr}\, A(a^\dagger) = \sum_m \langle m | A(a^\dagger) | m \rangle = \frac{1}{2\pi} \int_0^{2\pi} d\theta \langle \theta | A(a^\dagger) | \theta \rangle = \delta(0) \int_0^{2\pi} d\theta \, A(e^{i\theta}).
\tag{4.26}
$$

This implies that $\text{Tr}\, 1 = 2\pi\delta(0)$. Because the density in the $z$ direction of the D0-branes is $B$, we have $\text{Tr}\, 1 = \sum_m 1 = BL$. where $L$ is the IR cut off corresponds to the length of the $z$ direction. Then we find

$$\text{Tr}\, A(a^\dagger) = \frac{BL}{2\pi} \int_0^{2\pi} d\theta\, A(e^{i\theta}). \tag{4.27}$$

The momentum given in the matrix model is given by

$$\hat{P}_j = i[\hat{Z}_0, \hat{X}_0^j] = -i\frac{1}{B}\frac{\partial \tilde{X}^j(\hat{a}^\dagger)}{\partial \ln \hat{a}} = \frac{1}{B}\frac{\partial \tilde{X}^j(e^{i\theta})}{\partial \theta}\bigg|_{e^{-i\theta}\to\hat{a}}. \tag{4.28}$$

Thus the total momentum density $\frac{1}{L}\text{Tr}\,\hat{P}^j = \int_0^{2\pi} d\theta \frac{1}{2\pi}\frac{\partial \tilde{X}^j(e^{i\theta})}{\partial\theta}$ vanishes. The Hamiltonian of the matrix model for the supertube is [9]

$$\mathcal{H} = \frac{1}{2}\text{Tr}\sum_{j=1}^{8}\hat{P}_i^2 = L\frac{1}{B}\sum_{j=1}^{8}\frac{1}{2\pi}\int_0^{2\pi} d\theta \left(\frac{\partial \tilde{X}^j(e^{i\theta})}{\partial\theta}\right)^2, \tag{4.29}$$

where we dropped the constant term which is proportional to $B$. The fundamental string charge for the supertube in the matrix model is given by $\mathcal{H}/L$.

From the expression of the angular momentum in the matrix model,

$$\hat{L}_{ij} = \hat{X}_0^i \hat{P}_j - \hat{X}_0^j \hat{P}_i, \tag{4.30}$$

we find the total angular momentum density in the matrix model as

$$\frac{1}{L}\text{Tr}\,\hat{L}_{ij} = -\frac{1}{2\pi}\int_0^{2\pi} d\theta \left(\tilde{X}^i(e^{i\theta})\frac{\partial \tilde{X}^j(e^{i\theta})}{\partial\theta} - \tilde{X}^j(e^{i\theta})\frac{\partial \tilde{X}^i(e^{i\theta})}{\partial\theta}\right) \tag{4.31}$$

$$= -\frac{1}{\pi}\int_A d\tilde{X}^i \wedge d\tilde{X}^j. \tag{4.32}$$

In the section 4.3, we will compare these quantities with the corresponding ones for the supertube solutions in the D2-brane picture.

We have assumed $B$ is a constant, however, it has been known that even if $B$ depends on $\theta$ the supertube is a BPS state in the DBI action. In principle we can solve the zero-modes of the Dirac operator for this case and obtain the D0-brane picture. In this paper, however, we do not try to do it because of a technical difficulty. Instead, we will consider the supertube with the general cross section and a non-constant $B$ for the non-compact cross section in the section 5.1.

## 4.3 Comparison between the DBI and the matrix model descriptions

In this subsection, we will compare the various quantities of the supertube solution in the DBI description and the matrix model description. Here the DBI and the matrix model descriptions mean the D2-brane picture using the DBI action which ignore the derivative corrections and the D0-brane picture using the matrix model action which ignore the higher order terms, respectively. It it noted that the equivalence discussed in this paper use the

boundary state or boundary string field theory, thus the configuration in the D0-branes is exactly same as the configuration in the D2-brane. However, there will be differences in the two descriptions if we approximate it in the different ways.

First, we consider the D0-brane charge. We have seen that the D0-brane charge density along $z$ axis in the matrix model is $B$, which is not corrected by the higher order terms in the matrix model because it is essentially the size of the matrices. In the DBI picture, we can commutate the D0-brane charge from the Chern-Simons term as $\frac{1}{L}\frac{1}{2\pi}\int d\theta \int dz B = B$. Actually, these two computations should give same values since the Chern-Simons term may not be corrected by the derivative corrections.

However, the fundamental string charges will be different in two pictures. In the DBI description it is computed as [12]

$$q_{F1} = \frac{1}{L}\frac{1}{2\pi}\int d\theta \int dz \frac{1}{B}\sum_{j=1}^{8}(\frac{\partial X^j}{\partial \theta})^2 = \frac{1}{2\pi B}\int d\theta \sum_{j=1}^{8}(\frac{\partial X^j}{\partial \theta})^2 = \frac{R^2}{B}, \qquad (4.33)$$

for a constant $B$. On the other hand, the one computed in the matrix model is same form as (4.33), however, $X^j$ should be replaced by $\tilde{X}^j$ which was defined by (4.22). Thus, for example, the fundamental string charge in the matrix model for the circular supertube is $q_{F1} = e^{-\frac{1}{2BR}}R^2/B$. For the angular momentum, there is a similar difference between the two descriptions. These differences will be originated from the different approximations and will be absent if we includes all higher order and higher derivative corrections. We note that if the dimensionless parameter $BR$ is very large $BR \gg 1$, the differences become very small because $\tilde{X}^j \sim X^j$ for $BR \gg 1$.[9]

The higher order corrections for the matrix model action are expected to contain the commutators. If we assume the D0-brane action is given by the T-dual of the usual D$p$-brane actions, the corrections to the matrix model actions includes the terms corresponding to the DBI action, i,e, $[\hat{X},\hat{X}]^n \sim \mathcal{O}(\frac{R^n}{B^n})$, and terms corresponding to the higher derivative terms, i.e. $[\hat{X},[\hat{X},\cdots,[\hat{X},\hat{X}],],\cdots] \sim \mathcal{O}(\frac{1}{B^n})$. Thus the matrix model action may be reliable for $|B/R| \gg 1$ and $|B| \gg 1$. On the other hand, the DBI action may be reliable for $R \gg 1$. Thus $|BR| \gg 1$ is necessary for reliability of the both descriptions. We will discuss the ambiguity of the matrix model in subsection 5.3 again.

## 5. Some generalizations

### 5.1 Supertubes with non-compact cross section

Here we consider the flat super"tube". First, we consider the flat D2-brane with $(2\pi\alpha')F = Edt \wedge dy + Bdy \wedge dx$ in the flat spacetime. (Here $x$ and $y$ corresponds to the $\theta$ and $z$, respectively, in the previous section.) It is clear that this is a solution of the equations of motion for any constant $B, E$. Especially, For $E = 0$ this is the usual noncommutative D2-brane [34 − 36]. Below we will assume $B > 0$. The Dirac operator in a particular gauge

---

[9]For $BR \gg 1$, $\exp(-\frac{1}{2BR}) \sim 1 - \frac{1}{2BR} + \mathcal{O}(\frac{1}{B^2R^2})$ will be corrections to the DBI actions or the matrix model, but it is not $\alpha'$ correction, but like $1/N$ corrections.

is

$$i\not{D} = \sigma_x \left( \frac{\partial}{\partial x} - iBy \right) + \sigma_y \frac{\partial}{\partial y}. \tag{5.1}$$

In this case the zero-modes of the Dirac operator was already given in [5, 25] and the equivalent D0-brane picture was obtained. However, here we will use different basis of the zero-modes parametrised by a real number $\zeta$:

$$\Psi_\zeta(x,y) = \left( \frac{|B|^3}{4\pi^3} \right)^{\frac{1}{4}} \exp\left( iBy(x-\zeta) - \frac{|B|}{2}(x-\zeta)^2 \right) \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{5.2}$$

where $\Psi_\zeta$ is normalized such that,

$$\int dx dy \Psi_\zeta^\dagger \Psi_{\zeta'} = \delta(\zeta - \zeta'). \tag{5.3}$$

Then the matrix elements of $\hat{X} = x, \hat{Y} = y$ and $\hat{A}_t = -Ey$ between the zero-modes are computed as

$$\begin{aligned}
(\hat{X}_0)_{\zeta\zeta'} &= \int dx dy \Psi_\zeta^\dagger x \Psi_{\zeta'} = \zeta\delta(\zeta-\zeta'), \\
(\hat{Y}_0)_{\zeta\zeta'} &= \int dx dy \Psi_\zeta^\dagger y \Psi_{\zeta'} = \frac{1}{iB}\frac{\partial}{\partial\zeta'}\delta(\zeta-\zeta'), \\
(\hat{A}_{t,0})_{\zeta\zeta'} &= -E\,(\hat{Y}_0)_{\zeta\zeta'},
\end{aligned} \tag{5.4}$$

which can be represented in the Hilbert space spanned by $|x\rangle$, as $\hat{X}_0 = \hat{x}$ and $\hat{Y}_0 = \hat{y}$ where $\hat{x}|x\rangle = x|x\rangle$ and $[\hat{x},\hat{y}] = i/B$. This means that the D2-brane is equivalent to the infinitely many D0-branes with (5.4) as expected. We can also compute

$$\int dx dy\, \Psi_\zeta^\dagger\, e^{i\alpha x + i\beta y}\, \Psi_{\zeta'} = e^{-\frac{1}{4B}(\alpha^2+\beta^2+2i\alpha\beta)} e^{i\zeta'\alpha} \delta\left( \zeta - \zeta' + \frac{\beta}{B} \right). \tag{5.5}$$

This can be represented on $|x\rangle$ as

$$e^{-\frac{1}{4B}(\alpha^2+\beta^2+2i\alpha\beta)} e^{i\beta\hat{y}} e^{i\alpha\hat{x}} = e^{-\frac{1}{4B}(\alpha^2+\beta^2)} e^{i\alpha\hat{x}+i\beta\hat{y}} = e^{i\xi\hat{\eta}} e^{i\bar{\xi}\hat{\eta}^\dagger}, \tag{5.6}$$

where we have defined

$$\hat{\eta} \equiv \sqrt{\frac{B}{2}}(\hat{x}+i\hat{y}), \quad \xi \equiv \sqrt{\frac{1}{2B}}(\alpha-i\beta). \tag{5.7}$$

Note that $\hat{\eta}$ is a lowering operator

$$[\hat{\eta},\hat{\eta}^\dagger] = 1, \tag{5.8}$$

and (5.6) is anti-normal ordered. This result means that if the D2-brane has the nontrivial transverse coordinates $X^i = X^i(x,y)$ ($i = 2,3,\cdots,8$), the D0-branes have the transverse coordinates $\hat{X}_0^i = \left[ X^i(\hat{x},\hat{y}) \right]_A$ where $[\ldots]_A$ means the anti-normal ordering.

Now let us consider the non-compact supertube with an arbitrary magnetic field. It is the D2-brane with $F = Edt \wedge dy + B(x)dy \wedge dx$ and $X^i = X^i(x)$ ($i = 2,3,\cdots,8$) in the flat spacetime. Interestingly, for $E = \pm 1$ it was shown in [14] that this is indeed a BPS state

for all orders in the $\alpha'$ expansion. The coordinate $x$ can be chosen such that the Dirac operator becomes

$$i\not{D} = \sigma_x \left( \frac{\partial}{\partial x} - iBy \right) + \sigma_y \left( \frac{\partial}{\partial y} - iA_y(x) \right). \tag{5.9}$$

where we took a particular gauge for the magnetic field $B(x)$ such that $B(x) = B - \partial_x A_y(x)$ where $B$ is the constant part of $B(x)$ and, $A_0 = -Ey$, $X^i = X^i(x)$ $(i = 3, 4, \cdots, 9)$. Then the zero-modes are

$$\Psi_\zeta(x,y) = C_\zeta \exp\left( iBy(x - \zeta) - \frac{|B|}{2}(x - \zeta)^2 + F(x) \right) \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{5.10}$$

where

$$F(x) \equiv \int_0^x dx' A_y(x'), \tag{5.11}$$

and $C_\zeta$ is the normalization constant to be determined by $\int dx dy \Psi_\zeta^\dagger \Psi_{\zeta'} = \delta(\zeta - \zeta')$. The matrix elements are given by

$$(\hat{X}_0)_{\zeta\zeta'} = \int dx dy \Psi_\zeta^\dagger x \Psi_{\zeta'} = h(\zeta)\delta(\zeta - \zeta'),$$

$$(\hat{Y}_0)_{\zeta\zeta'} = \int dx dy \Psi_\zeta^\dagger y \Psi_{\zeta'} = \frac{1}{iB}\frac{\partial}{\partial \zeta'}\delta(\zeta - \zeta'), \quad (\hat{A}_{t,0})_{\zeta\zeta'} = -E\,(\hat{Y}_0)_{\zeta\zeta'},$$

$$(\hat{X}_0^i)_{\zeta\zeta'} = \int dx dy \Psi_\zeta^\dagger X^i(x)\Psi_{\zeta'} = h^i(\zeta)\delta(\zeta - \zeta') \tag{5.12}$$

where

$$h(\zeta) \equiv \zeta + \frac{\int dx\, x\, e^{-|B|x^2 + 2F(x+\zeta)}}{\int dx\, e^{-|B|x^2 + 2F(x+\zeta)}}, \quad h^i(\zeta) \equiv \frac{\int dx\, X^i(x) e^{-|B|(x-\zeta)^2 + 2F(x)}}{\int dx\, e^{-|B|(x-\zeta)^2 + 2F(x)}}. \tag{5.13}$$

These can be represented as $\hat{X}_0 = h(\hat{x})$, $\hat{Y}_0 = \hat{y}$ and $\hat{X}_0^i = h^i(\hat{x})$ and indeed satisfy the BPS equations of the matrix model.[10]

In the appendix, we will consider the inverse of the transformation, i.e. the D2-brane supertube from the supertube in the matrix model using infinitely many D2-brane-anti-D2-brane pairs.

## 5.2 Superfunnel type solutions

In this subsection, instead of $A_t = -Ey$, we take $A_t = -EZ$ where $E = \pm 1$. Then for $\triangle z(x,y) = 0$ and $x^i = 0$ $(i = 3, \cdots, 8)$, the D2-brane is the noncommutative BIon, or the superfunnel. The zero-modes of the tachyon are same as in the previous subsection. Thus using (5.5) and (5.6), the D0-branes have the transverse coordinates $\hat{Z}_0 = [z(\hat{x}, \hat{y})]_A$, where $[\ldots]_A$ means the anti-normal ordering. This satisfies the equations of motion and the Gauss law of the matrix model (3.9) because the derivation keeps the anti-normal ordering.

---

[10]For the pure gauge $F(x) = ax + c$, we have $\hat{X}_0 = \hat{x} - a/B$. We can redefine $\hat{x}' = \hat{x} - a/B$ which does not change the commutation relation. For $F(x) = -bx^2/2$, we have $\hat{X}_0 = \frac{B}{B+b}\hat{x}$, which means the shift of $B$ to $B + b$ because $[\hat{X}_0, \hat{Y}_0] = i/(B + b)$. This is consistent with the fact that $F(x) = -bx^2/2$ is the shift of $B$.

The D2$p$-brane with $F = Edt \wedge dz + b_a dy^a \wedge dx^a$ $(a = 1, \cdots, p)$ where $E = \pm 1$, $\triangle z(x_a, y_a) = 0$ and $x^i = 0$ $(i = 2p+1, \cdots, 8)$, is the noncommutative BIon, which is the solution of the equation of motion of the $N = 4$ supersymmetric Yang-Mills actions. The equivalent D0-branes are easily obtained as before, and we have $\hat{Z}_0 = [z(\hat{x}^a, \hat{y}^a)]_A$ although the solution of the equations of motion of the matrix model

$$\sum_{a=1}^{p} \frac{1}{b_a^2} \left( \frac{\partial^2}{\partial^2 \hat{X}^a} + \frac{\partial^2}{\partial^2 \hat{Y}^a} \right) \hat{Z}(\hat{X}, \hat{Y}) = 0, \tag{5.14}$$

is different from $\triangle z(x_a, y_a) = 0$ except when all $b_a$s are same. However, in this case, we do not know if the equations of motion should be corrected beyond the Yang-Mills or the matrix model approximation. Note that the former is valid for $|b_a| \ll 1$ and $r \equiv \sqrt{(x^a)^2 + (y^a)^2} \gg 1$ and the latter is valid for $|b_a| \gg 1$ and $r \gg 1$. Moreover, it is possible that two configurations can be matched by some field redefinition. Indeed, by the field redefinition of the theory on the D($2p$)-brane, we can use the noncommutative Yang-Mills actions with $\theta = 1/B$, $\Phi = -B$ and $G = -B\frac{1}{g}B$, i.e. $S \sim \sqrt{G}G^{-1}((\hat{F}_* - B))^2$ which was shown to be equivalent to the matrix model action [33]. Then, the equations of motion of the noncommutative Yang-Mills action is same as (5.14) because of the $G^{-1} \sim 1/|b^a|$ factor. This is natural since the string world renormalization conditions for the D2-brane and the D0-branes will be same for this choice of the fields as seen from [33].[11]

## 5.3 Comments on the ambiguity of the effective action of the D0-branes

In the previous sections, we take the effective action of the D0-branes as the matrix models with the higher order corrections derived from the T-dual, i.e. the dimensional reduction of the DBI actions.[12] However, the effective actions of the D-branes have the field redefinition ambiguity, which is interpreted as the regularization (or, more precisely, the finite renormalization) of the string world sheet action, in general. Therefore, in order to compare the physical quantities in the D0-brane and D($2p$)-brane pictures, we should use the effective actions obtained from the string world sheet computations with the same regularization, or more precisely the same renormalization condition, even though the two boundary states in the D0-brane and D($2p$)-brane pictures are equivalent and every quantities should be same in principle.[13]

---

[11]In [33] the Weyl ordered product and the Moyal star product are used. However, any ordering can be used in the matrix model description of the noncommutative gauge theory, especially the anti-normal ordered product. Actually, it can be shown as in [33] that the equivalence between the matrix model with the anti-normal ordering and the noncommutative gauge theory of $\Phi = -B$ with the star product of the anti-normal ordering. Note that the choice of the ordering in the matrix model does not change the operator, for example $\hat{A}_i(\hat{x})$, though we should redefine the noncommutative gauge field in the noncommutative gauge theory side.

[12]For the non-Abelian case, there are ordering ambiguities, which is, however, can be fixed by the string world sheet computation in principle.

[13]We would have to choose some regularization such that the reduction to the zero-modes of the Dirac operator are valid after the regularization and the equivalence between the path-integral and the operator formalism used in [28] to show the equivalence between D-brane systems is still valid.

The effective action of the D0-branes given by the T-dual are based on the very special choice of the regularization of the world sheet although it is a natural choice. For the flat noncommutative D(2p)-brane solution in the D0-branes, this choice corresponds to the the noncommutative parameter $\theta = 1/B$ and $\Phi = -B$ as seen from the results of [33]. However, if we make the following field redefinition, $\hat{X}^i \rightarrow \hat{X}^i + \beta^{kl}[\hat{X}^k, \hat{X}^l]\hat{X}^i$, we have a different noncommutative parameter $\theta \equiv [\hat{X}^i, \hat{X}^j] = (1/B)_{ij} \rightarrow ((1 + \beta/B)^2/B)_{ij}$. Note that the $U(N)$ symmetry of the matrix model should contain the noncommutative $U(1)$ gauge symmetry of the D(2p)-brane with any noncommutative parameter $\theta$, including usual commutative gauge symmetry since there are the Seiberg-Witten maps in the D(2p)-brane effective actions.

From this observation of the noncommutative D(2p)-brane, we conclude that we have to refine the fields, say $\hat{X}^i$, to compare the physical quantities computed using the usual commutative DBI action of the D2p-brane and the matrix model of the D0-branes derived from the the dimensional reduction of the DBI action.

We also note that there should be some differences in the renormalization conditions between the computations directly using the result of [3] and ours using the D0$-\overline{\text{D0}}$-brane pairs.

## 5.4 Comments for the Fuzzy $S^2$ case

Finally, we will briefly comment on the fuzzy $S^2$ case comparing to the supertube case. In [5], the D2-brane on the fuzzy $S^2$ was considered using the tachyon condensation in the same way in this paper. In the fuzzy $S^2$ case, the DBI action would be reliable for $R \gg 1$ (or $R \gg l_s$ if we recover the $\alpha'$ dependence), but the matrix model would be reliable for $1/M_{D0} \gg 1$ (or $1/M_{D0} \gg l_s$) where $1/M_{D0} \equiv N l_s^2/R$.[14] Moreover, $N = RM_{D0}$ (which is a dimensionless parameter) controls the similarity of two descriptions, i.e. in the $N \rightarrow \infty$ limit the two descriptions are same. We have seen that the parameter $BR$ for the supertube plays a similar role as $N$ of the fuzzy $S^2$ case. Actually, $BR$ is approximately the number of the D0-branes in a sphere of radius $R$ centered at a point on $Z$ axis.

Here we note that $N = 1$ and $N = 0$ are special. The matrix model is always reliable for these cases because $[\hat{X}^i, \hat{X}^j] = 0$. This $N = 1$ case is like the supertube with $BR = 0$ in which the effective radius is infinitely small. (This $N = 1$ case is also similar to the tachyon condensation of two non BPS D0-branes to a non BPS D0-brane in [5].) Furthermore, if we take $R \gg 1$ for these, both the matrix model and the DBI descriptions seem good although these two descriptions are apparently different. However, the boundary state with the path-integral over the curved world volume used in [5] might be inappropriate for these cases. Indeed, if we take very large, but finite $u$, the D2-brane is localized on $S^2$ with the radius $R$.[15]

---

[14]Here $R$ is the radius of the $S^2$ and $N$ is the number of the D0-branes.

[15]Here the localized brane means localized in the usual sense, for example, the supertube with $R \gg 1$ and $|B| \ll 1$. In [5] the noncommutativity means the noncommutativity in the D0-picture, i.e. $[\hat{X}^i, \hat{X}^j] \neq 0$, and the localized branes means the branes which is commutative in the D0-picture. However, the localized brane in the usual sense can be "non-localized" in the D0-picture. Thus the term "localized" in [5] is not appropriate in the usual sense.

In order to see how the fuzzy $S^2$ in an on-shell configuration behaves, let us consider the $N$ monopoles in the D3-branes [6]. Here $N < 0$ case corresponds to $|N|$ anti-monopoles. The radius of the fuzzy sphere is $R \sim |N|/(\xi \pm a/2)$ where $\xi$ is the coordinate along the D1-branes and D3-branes are at $\xi \pm a/2$. The D1-brane picture is good for $|N|/R \gg 1$, on the other hand, the D3-brane picture is good for $R \gg 1$ [37]. [16] For $|N| = 1$, the D1-brane can not form the fuzzy sphere. Nevertheless the D1-brane picture will be valid for $a \gg 1$ and $|\xi \pm a/2| \gg 1$ which means $R \ll 1$. Near the boundary $\xi \sim \pm a/2$, the D1-brane picture for $|N| = 1$ is singular, but the D3-brane picture is valid. (More precisely, for $|N| = 1$ the singularities at the boundaries are worse than $|N| > 1$ case. ) Two pictures should be smoothly connected. In this case we took $u \to \infty$ limit, but the D3-brane picture is valid near $|\xi \pm a/2| \sim 0$ because the D1-brane picture should be corrected near the singularity $|\xi \pm a/2| \sim 0$. Thus it is interesting to see if the difference of the radii of the supertubes with the compact cross section in the two pictures is an artifact or not. We note that in the monopole case, although the embedding into the flat spacetime is nontrivial, we have not used the path-integral over the curved world volume.

## Acknowledgments

## A. The inverse transformation

In [6, 7], both of the ADHM(N) construction of instanton (monopole) and the inverse ADHM(N) construction were obtained using the tachyon condensation. In this appendix we will find the inverse transformation for the supertube with the non-compact cross section. Here the inverse transformation for the supertube means that the construction of the supertube from the infinitely many D2-brane and anti-D2-brane pairs instead of the D0$-\overline{\text{D0}}$-branes.

First, let us remember that using the decent relations [20] a D0-brane at $x^1 = a^1, x^2 = a^2$ can be constructed from a pair of the D2-brane and the anti-D2-brane by giving $T = u\sigma_1(x^1 - a^1) + \sigma_2(x^2 - a^2))$ and $A_1 = A_2 = 0$ [23, 24]. Thus, we can obtain the infinitely many D0-branes with $\hat{X}^i = \hat{x}^i$ where $[\hat{x}^1, \hat{x}^2] = i/B$ from infinitely many pairs of the D2-brane and the anti-D2-brane with

$$T(x^1, x^2) = u(\sigma_1(x^1 - \hat{x}^1) + \sigma_2(x^2 - \hat{x}^2)) \tag{A.1}$$

and $A_1 = A_2 = 0$. Furthermore, $\hat{A}_t = E\hat{x}^2$ is induced in the D0-brane picture by setting $A_t = E\hat{x}^2$ for the pairs of the D2-brane and the anti-D2-brane.

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[16] For the $k$ instantons, the D4-brane picture is valid if a scale of the instantons $\rho$ is very large. On the other hand, D0-brane picture is valid if the mass scale of the D0-branes $M$ is large and there is a relation $\rho M \sim |k|$.

Then, to obtain the D2-brane picture, we should solve

$$\not{D}\psi(z) = \begin{pmatrix} 0 & \bar{z} - \hat{z}^\dagger \\ z - \hat{z} & 0 \end{pmatrix} \begin{pmatrix} \psi_1(z) \\ \psi_2(z) \end{pmatrix} = 0, \tag{A.2}$$

where $z = x^1 + ix^2$ and $\hat{z} = \hat{x}^1 + i\hat{x}^2$. There is only one normalizable zero-mode of the tachyon which corresponds to the surviving D2-brane after the tachyon condensation:

$$\psi_1(z) = |z\rangle, \quad \psi_2 = 0, \tag{A.3}$$

where $|z\rangle$ is the normalized coherent state $\hat{z}|z\rangle = z|z\rangle$. This zero-mode depends on $z$, thus we have [6]

$$A_z = 2i\psi^\dagger D_{\bar{z}}\psi = -i\frac{B}{2}z, \tag{A.4}$$

where $A_z = A_1 + iA_2$ and $D_z = \frac{1}{2}(\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2})$. Note that it can be considered as the Berry phase although it is not an approximation, but an exact result in our $u \to \infty$ limit. Thus we have the D2-brane with $F_{12} = -B$. For the time component of the gauge field we have

$$A_t = \psi^\dagger E\hat{x}^2\psi = Ex^2. \tag{A.5}$$

Thus we have shown the equivalence by giving the map from the D0-brane picture to the D2-brane picture.

We can also consider the non-compact supertube in the D0-brane picture with the nontrivial transverse coordinate

$$\hat{X}^i = \int d\alpha^i \int d\beta^i \, C(\alpha^i, \beta^i) \, e^{i\bar{\xi}^i\hat{\eta}^\dagger} e^{i\xi^i\hat{\eta}}, \tag{A.6}$$

where $\xi^i = \sqrt{\frac{1}{2B}}(\alpha^i - i\beta^i)$ and $\hat{\eta} \equiv \sqrt{\frac{B}{2}}(\hat{x}^1 + i\hat{x}^2)$. Then in the D2-brane picture it becomes

$$X^i(x^1, x^2) = \int d\alpha^i \int d\beta^i \, C(\alpha^i, \beta^i) \, e^{i\alpha^i x^1 + i\beta^i x^2}. \tag{A.7}$$

Here we note that (A.6) is normal ordered instead of anti-normal ordered used in (5.6). This difference could be originated from the possible field redefinition.

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